

Farey series and their connection with the prime number problem. I.

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Let $x \geq 1$; we denote by F_x the ascending sequence of fractions $\frac{k}{n}$ (FAREY series of order x) for which

$$0 < k \leq n \leq x, \quad (k, n) = 1.$$

The ν -th term of F_x will be denoted by ρ_ν ; the number of these fractions is

$$\Phi(x) = \sum_{n=1}^{[x]} \varphi(n),$$

$\varphi(n)$ denoting EULER's function.

It is well-known that the so-called FAREY dissection of the continuum is a very important tool in the additive theory of numbers; the sphere of applications extended still more when it was discovered that the equidistribution of the FAREY series is connected with the validity of RIEMANN's hypothesis (i. e. with the assumption that the zeta-function of RIEMANN has no roots for $\Re(s) > \frac{1}{2}$).

In the first place, it has been proved by J. E. LITTLEWOOD¹⁾ that RIEMANN's hypothesis is true if and only if the relation

$$M(x) = \sum_{n=1}^{[x]} \mu(n) = \sum_{\nu=1}^{\Phi(x)} \cos 2\pi \rho_\nu = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\mu(n)$ denotes the function of MÖBIUS, holds for all positive values of ε .

The later result of J. FRÄNEL²⁾, which is mentioned by E. LANDAU as "ein hübscher Satz", can be expressed as follows:

Let us divide the interval $\langle 0, 1 \rangle$ into $\Phi(x)$ equal parts, furthermore let us mark the fractions of F_x ; we then form the sum of squares

¹⁾ [1], 263–266; see LANDAU [2], vol. II, 161–166. (Numbers in brackets [] refer to the bibliography placed at the end of the paper.)

²⁾ [1], 198–201; see LANDAU [2], vol. II, 167–177.

of the differences

$$\delta_\nu = \delta_\nu(x) = \varrho_\nu - \frac{\nu}{\Phi(x)}, \quad \nu = 1, 2, \dots, \Phi(x).$$

RIEMANN'S hypothesis is equivalent to the assertion that

$$\sum_{\nu=1}^{\Phi(x)} \delta_\nu^2 = O(x^{-1+\varepsilon})$$

or (as LANDAU added)³⁾

$$\sum_{\nu=1}^{\Phi(x)} |\delta_\nu| = O(x^{\frac{1}{2}+\varepsilon}),$$

ε being any positive number.

Now, we consider in this paper the asymptotical behaviour for $x \rightarrow \infty$ of the sums of type

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu),$$

$f(t)$ denoting a function which is defined at the points $0 < \varrho_\nu \leq 1$ ($\nu = 1, 2, \dots, \Phi(x)$) and belongs to a class as wide as possible.

By supposing that $f(t)$ is bounded; integrable (in RIEMANN'S sense) for $0 \leq t \leq 1$, it follows from the uniform distribution of the fractions of F_x ($x \rightarrow \infty$) that

$$(I) \quad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim \Phi(x) \int_0^1 f(t) dt;$$

in § 1 we shall show that (I) holds certainly if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right)$$

exists, or if $f(t)$ is continuous, decreasing, non-negative for $0 < t \leq 1$ and

$$\lim_{\varepsilon \rightarrow +0} \int_\varepsilon^1 f(t) dt$$

exists.

To find a better bound for the difference

$$R_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t) dt,$$

³⁾ See LANDAU [3], 202–206. We mention herewith a later result of LANDAU the relation

$$\max_{1 \leq \xi \leq \Phi(x)} \left| \sum_{\nu=1}^{[\xi]} \delta_\nu \right| = O\left(x^{\frac{1}{9}+\varepsilon}\right)$$

is also equivalent to RIEMANN'S hypothesis. ([4], 347–352.)

we may suppose that $f(t)$ has a bounded derivative in the interval $\langle 0, 1 \rangle$; it will be proved that in this (particularly important) case the behaviour of $R_r(x)$, concerning its order of magnitude, is the same as that of $M(x) = \sum_{n \leq x} \mu(n)$ or of the remainder of the prime number theorem.

$$\pi(x) - \int_2^x \frac{du}{\log u};$$

we have namely, according to § 2,

$$(II) \quad R_r(x) = O(xe^{-c_3(\log x)^\gamma})$$

where γ is any constant between $\frac{1}{2}$, $\frac{11}{21}$ excl. and $c_3 > 0$ depends on the choice of γ only; by supposing the validity of RIEMANN'S hypothesis, it follows that the much sharper relation

$$(III) \quad R_r(x) = O\left(x^{\frac{1}{2} + c_6 \frac{\log \log \log x}{\log \log x}}\right) = O\left(x^{\frac{1}{2} + \epsilon}\right),$$

c_6 denoting another positive constant⁴), holds also.

For a function $f(t)$ which has continuous derivatives $f'(t), f''(t), \dots, f^{(2r+1)}(t)$, the "EULER—MACLAURIN sum-formula" furnishes explicitly

$$(IV) \quad \sum_{v=1}^{\Phi(x)} f(\varrho_v) = \Phi(x) \int_0^1 f(t) dt + \frac{1}{2} (f(1) - f(0)) + \\ + \frac{B_2}{2!} (f'(1) - f'(0)) \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right) + \\ + \frac{B_4}{4!} (f'''(1) - f'''(0)) \sum_{n=1}^{[x]} \frac{1}{n^3} M\left(\frac{x}{n}\right) + \dots + \\ + \frac{B_{2r}}{(2r)!} (f^{(2r-1)}(1) - f^{(2r-1)}(0)) \sum_{n=1}^{[x]} \frac{1}{n^{2r-1}} M\left(\frac{x}{n}\right) + U_r(x),$$

where

$$-U_r^2(x) = \vartheta \frac{B_{4r+2}}{(4r+2)!} \int_0^1 (f^{(2r-1)}(t))^2 dt \sum_{a,b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a,b)^{4r+2}}{a^{4r+1} b^{4r+1}} \\ (0 \leq \vartheta = \vartheta(x, r) \leq 1),$$

B_2, B_4, B_6, \dots being the well-known numbers of BERNOULLI.

In case of $r=0$ we have in particular

$$(V) \quad \sum_{v=1}^{\Phi(x)} f(\varrho_v) = \Phi(x) \int_0^1 f(t) dt + \frac{1}{2} (f(1) - f(0)) + \\ + \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) \int_0^1 \left(nt - [nt] - \frac{1}{2}\right) f'(t) dt;$$

⁴) Here and later ϵ denotes an arbitrary positive number.

the last term on the right-hand side may be replaced (on the basis of the so-called FRANEL-identity) by the positive square root of

$$\vartheta \int_0^1 (f'(t))^2 dt \left(\frac{1}{12} + \Phi(x) \sum_{v=1}^{\Phi(x)} \delta_v^2 \right) \quad (0 \leq \vartheta = \vartheta(x) \leq 1).$$

We obtain rather deep results by discussing the question as follows (§ 3): when does the converse of (III) hold also, i. e. for which functions $f(t)$, having a bounded derivative in $\langle 0, 1 \rangle$, does the validity of RIEMANN's hypothesis follow from the existence of a relation of type (III)?

It can be simply proved:

Let $\lambda \geq \frac{1}{2}$; RIEMANN's hypothesis is true if and only if we have for every positive ε

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

By using this lemma, the formulae (IV), (V) and some relations from the theory of DIRICHLET series respectively, we get the following interesting theorems:

1) For any polynomial of second degree

$$f(t) \equiv a_0 t^2 + a_1 t + a_2 \quad (a_0 \neq 0)$$

the relation

$$(VI) \quad R_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

holding of every $\varepsilon > 0$, is equivalent to RIEMANN's hypothesis.

2) Let us take

$$f(t) \equiv a_0 t^3 + a_1 t^2 + a_2 t + a_3 \quad (a_0 \neq 0),$$

then a necessary and sufficient condition that (VI) should be equivalent to RIEMANN's hypothesis is that

$$a_1 \neq -\frac{3}{2} a_0.$$

3) For every $r \geq 2$, there is an infinity of polynomials

$$f(t) \equiv a_0 t^r + a_1 t^{r-1} + \dots + a_{r-1} t + a_r \quad (a_0 \neq 0)$$

such that (VI) be equivalent to RIEMANN's hypothesis.

4) Suppose that $f(t)$ is defined and has continuous derivatives $f'(t), f''(t), f'''(t) \neq 0$ for $0 \leq t \leq 1$, furthermore the condition

$$\frac{|f'(1) - f'(0)|}{\int_0^1 |f''(t)| dt} > \frac{3\zeta(3)}{2\pi} = 0.574 \dots$$

should be satisfied. Then (VI) is equivalent to RIEMANN's hypothesis.

In particular, we have the simple relations (found to be equivalent to the hypothesis in question)

$$\sum_{v=1}^{\Phi(x)} \varrho_v^2 - \frac{\Phi(x)}{3} = O\left(x^{\frac{1}{2}+\varepsilon}\right), \quad \sum_{v=1}^{\Phi(x)} \varrho_v^3 - \frac{\Phi(x)}{4} = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

$$\sum_{v=1}^{\Phi(x)} \cos \lambda \varrho_v - \frac{\sin \lambda}{\lambda} \Phi(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad \text{with } 0 < \lambda \leq \frac{\pi}{2}.$$

These results show very clearly, how close the connection is between the problem of the distribution of FAREY fractions (i. e., a problem of elementary number theory) and that of the situation of "zeta-roots" in the theory of functions.

1. Uniform distribution and its consequences.

Let x be a positive variable, and let a, b, k, l, m, n, r, v denote throughout positive integers.

We begin by some fundamental identities.

Lemma 1. *We have for any function $f(t)$ which is defined at the points $t = \frac{k}{n}$ ($k = 1, 2, \dots, n$)*

$$\sum_{\substack{k=1 \\ (k, n)=1}}^n f\left(\frac{k}{n}\right) = \sum_{d|n} \mu(d) \sum_{k=1}^{\frac{n}{d}} f\left(d \frac{k}{n}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{k=1}^d f\left(\frac{k}{d}\right),$$

where $d|n$ means that the summation is extended over all divisors of n .

Proof: It is evident that

$$\sum_{d|n} \sum_{\substack{k=1 \\ (k, d)=1}}^d f\left(\frac{k}{d}\right) = \sum_{k=1}^n f\left(\frac{k}{n}\right)$$

and our assertion follows by MÖBIUS inversion.

Lemma 2. *Let $f(n)$ and $g(n)$ be arbitrary arithmetical functions. Then we have*

$$(1) \quad \sum_{n=1}^{[x]} \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d=1}^{[x]} f(d) \sum_{\delta=1}^{\left[\frac{x}{d}\right]} g(\delta) = \sum_{d=1}^{[x]} g(d) \sum_{\delta=1}^{\left[\frac{x}{d}\right]} f(\delta),$$

in particular

$$(2) \quad \sum_{n=1}^{[x]} \sum_{d|n} f(d) = \sum_{d=1}^{[x]} \left[\frac{x}{d} \right] f(d) = \sum_{d=1}^{[x]} \sum_{\delta=1}^{\left[\frac{x}{d}\right]} f(\delta).$$

Proof: We write

$$(3) \quad \sum_{n=1}^{[x]} \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{\substack{d, \delta \\ d\delta \leq x}} f(d) g(\delta) = \\ = f(1) \sum_{\delta=1}^{[x]} g(\delta) + f(2) \sum_{\delta=1}^{\left[\frac{x}{2}\right]} g(\delta) + \dots = \sum_{d=1}^{[x]} f(d) \sum_{\delta=1}^{\left[\frac{x}{d}\right]} g(\delta).$$

On the other hand, by

$$\sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d)$$

$f(n)$ and $g(n)$ may be exchanged under (3).

The combination of Lemma 1 and Lemma 2 (with $f(n) \equiv \mu(n)$, $g(n) \equiv \sum_{k=1}^n f\left(\frac{k}{n}\right)$) gives

$$\text{Lemma 3. Let us take } M(x) = \sum_{n=1}^{[x]} \mu(n), \quad V(n) = \sum_{k=1}^n f\left(\frac{k}{n}\right),$$

then

$$(4) \quad \sum_{\nu=1}^{Q(x)} f(q_\nu) = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) V(n) = \sum_{n=1}^{[x]} \mu(n) \sum_{d=1}^{\left[\frac{x}{n}\right]} V(d),$$

provided that $f(t)$ is defined for the values in question of its argument.

We shall see that the identities (4) are useful in order to find asymptotic formulae for $\sum f(q_\nu)$.

Lemma 4. Let $0 \leq \xi \leq 1$ and let us denote by $h(\xi, x)$ the number of fractions in F_x which are not greater than ξ . Then we have

$$h(\xi, x) = \sum_{\substack{p \leq x \\ p \leq \xi x}} 1 = \sum_{n=1}^{[x]} [n\xi] M\left(\frac{x}{n}\right) = \sum_{n=1}^{[x]} \mu(n) \sum_{d=1}^{\left[\frac{x}{n}\right]} [d\xi].$$

Proof: Using the fundamental property of the MÖBIUS function we get

$$\sum_{\substack{k \leq \xi x \\ (k, n)=1}} 1 = \sum_{k=1}^{[\xi n]} \sum_{d|(k, n)} \mu(d) = \sum_{d|n} \mu(d) \left[\frac{\xi n}{d}\right] = \sum_{d|n} \mu\left(\frac{n}{d}\right) [d\xi],$$

so that (1) furnishes indeed

$$h(\xi, x) = \sum_{n=1}^{[x]} \sum_{d|n} [d\xi] \mu\left(\frac{n}{d}\right) = \sum_{d=1}^{[x]} [d\xi] M\left(\frac{x}{d}\right) = \sum_{d=1}^{[x]} \mu(d) \sum_{\delta=1}^{\left[\frac{x}{d}\right]} [\delta\xi].$$

In what follows we need also the familiar identities, arising immediately from (1),

$$(5) \quad \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) = \sum_{n=1}^{[x]} \mu(n) \left[\frac{x}{n}\right] = 1,$$

$$(6) \quad \Phi(x) = \sum_{n=1}^{[x]} \varphi(n) = \sum_{n=1}^{[x]} n M\left(\frac{x}{n}\right) = \frac{1}{2} \sum_{n=1}^{[x]} \mu(n) \left[\frac{x}{n}\right]^2 + \frac{1}{2}.$$

By use of (6) it is easy to show that⁵⁾

$$(7) \quad \Phi(x) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Next we make use of a well-known result of H. WEYL⁶⁾:

If t_n ($n=1, 2, 3, \dots$) is a sequence such that $(t_n) = t_n - [t_n]$ ($n=1, 2, 3, \dots$) is uniformly distributed in $\langle 0, 1 \rangle$, then we have for all RIEMANN integrable functions $f(t)$ with the period 1

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(t_k) = \int_0^1 f(t) dt, \quad \sum_{k=1}^n f(t_k) \sim n \int_0^1 f(t) dt.$$

In order to apply this proposition we prove

Theorem 1. *The distribution of F_x becomes uniform when $x \rightarrow \infty$.*

Proof: Consider the sequence

$$(8) \quad \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots$$

Let $0 \leq \xi_1 \leq \xi_2 \leq 1$ and suppose that the denominator of the n -th term under (8) is $[x] + 1$. Then we have (cf. (7))

$$n = \Phi(x) + \vartheta x = \frac{3}{\pi^2} x^2 + O(x \log x) \quad (0 \leq \vartheta \leq 1),$$

on the other hand, the number of fractions among the first n terms which lie in the interval $\langle \xi_1, \xi_2 \rangle$ is plainly

$$n_{\xi_1, \xi_2} = n_{\xi} = h(\xi_2, x) - h(\xi_1, x) + \Theta x \quad (0 \leq \Theta \leq 1).$$

Thus, by Lemma 4, (6) and (7), we can write

$$\begin{aligned} n_{\xi} &= \sum_{d=1}^{[x]} ([d\xi_2] - [d\xi_1]) M\left(\frac{x}{d}\right) + \Theta x \\ &= (\xi_2 - \xi_1) \Phi(x) + O\left(x \sum_{d=1}^{[x]} \frac{1}{d}\right) + O(x) = (\xi_2 - \xi_1) n + O(x \log x) \end{aligned}$$

whence it follows indeed that

$$\frac{n_{\xi}}{n} = (\xi_2 - \xi_1) + \frac{O(x \log x)}{n} \rightarrow \xi_2 - \xi_1$$

when $n \rightarrow \infty$, i. e. $x \rightarrow \infty$.

⁵⁾ See e. g. HARDY-WRIGHT [1], 266.

⁶⁾ [1], 314.

The above result of WEYL being applicable to FAREY fractions, we obtain that

$$(9) \quad \sum_{v=1}^{\Phi(x)} f(\varrho_v) \sim A \Phi(x) \sim \frac{3A}{\pi^2} x^2 \quad \text{with} \quad A = \int_0^1 f(t) dt,$$

if $f(t)$ is a bounded, RIEMANN integrable function in $\langle 0, 1 \rangle$.

The validity of (9) can be proved under more general conditions, by using the following theorem of TOEPLITZ⁷⁾:

Let $t_1, t_2, \dots, t_n, \dots$ be a convergent sequence with the limit zero and suppose that the numbers a_{kl} ($k, l = 1, 2, 3, \dots$) satisfy the following conditions:

- 1) for any fixed l , $a_{kl} \rightarrow 0$ when $k \rightarrow \infty$,
- 2) $S(k) = |a_{k1}| + |a_{k2}| + \dots + |a_{kk}| = O(1)$.

Then the sequence

$$t'_k = a_{k1}t_1 + a_{k2}t_2 + \dots + a_{kk}t_k \quad (k = 1, 2, \dots)$$

converges also to zero.

Theorem 2. Let $f(t)$ be a function defined at all rational points of the interval $0 < t \leq 1$ ⁸⁾, such that

$$\frac{V(n)}{n} = \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right)$$

converges when $n \rightarrow \infty$, and has the limit A . Then we have

$$\sum_{v=1}^{\Phi(x)} f(\varrho_v) \sim A \Phi(x) \sim \frac{3A}{\pi^2} x^2.$$

Proof: In virtue of (4) and (6) we need only to show that, if our conditions are satisfied, then

$$\frac{1}{\Phi(x)} \sum_{v=1}^{\Phi(x)} f(\varrho_v) - A = \frac{1}{\Phi(x)} \sum_{n=1}^{[x]} n M\left(\frac{x}{n}\right) \left(\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) - A \right) \rightarrow 0,$$

when $x \rightarrow \infty$.

But this follows therefrom that, ex hypothesi,

$$\frac{V(n)}{n} - A \rightarrow 0,$$

moreover (cf. (7)),

- 1) for any fixed n

$$\frac{n \left| M\left(\frac{x}{n}\right) \right|}{\Phi(x)} \leq \frac{x}{\Phi(x)} \sim \frac{\pi^2}{3x} \rightarrow 0 \quad \text{when } x \rightarrow \infty,$$

⁷⁾ See e. g. KNOPP [1], 75.

⁸⁾ Observe that the point 0 does not belong to the interval.

$$2) S(x) = \frac{1}{\Phi(x)} \sum_{n=1}^{\lfloor x \rfloor} n \left| M\left(\frac{x}{n}\right) \right| \leq \frac{x^2}{\Phi(x)} \sim \frac{\pi^2}{3},$$

and so TOEPLITZ's theorem may be applied.

In certain cases it is somewhat more convenient to use the following

Corollary. *If $f(t)$ is continuous, decreasing (if t increases), non-negative for $0 < t \leq 1$, and if*

$$\lim_{\varepsilon \rightarrow +\infty} \int_{\varepsilon}^1 f(t) dt = \int_0^1 f(t) dt$$

exists, then we have

$$\sum_{v=1}^{\Phi(x)} f(\rho_v) \sim \Phi(x) \int_0^1 f(t) dt.$$

Proof: Suppose that $f(t)$ satisfies our conditions.

We write

$$\begin{aligned} \sum_{k=1}^n f\left(\frac{k}{n}\right) - \int_1^n f\left(\frac{u}{n}\right) du &= \sum_{k=1}^{n-1} \left(f\left(\frac{k}{n}\right) - \int_k^{k+1} f\left(\frac{u}{n}\right) du \right) + f(1) = \\ &= \sum_{k=1}^{n-1} \int_k^{k+1} \left(f\left(\frac{k}{n}\right) - f\left(\frac{u}{n}\right) \right) du + f(1), \end{aligned}$$

and hence, considering that

$$0 \leq \int_k^{k+1} \left(f\left(\frac{k}{n}\right) - f\left(\frac{u}{n}\right) \right) du \leq f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n}\right),$$

it follows

$$(10) \quad \sum_{k=1}^n f\left(\frac{k}{n}\right) - \int_1^n f\left(\frac{u}{n}\right) du \leq \sum_{k=1}^{n-1} \left(f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n}\right) \right) + f(1) = f\left(\frac{1}{n}\right).$$

Since

$$\int_{\frac{1}{n}}^1 f(t) dt = - \int_n^1 f\left(\frac{1}{v}\right) \cdot \frac{1}{v^2} dv = \int_1^n \frac{f\left(\frac{1}{v}\right)}{v^2} dv,$$

the integral

$$\int_1^\infty \frac{f\left(\frac{1}{v}\right)}{v^2} dv$$

exists by hypothesis; there exists therefore to any $\varepsilon > 0$ a number

$N = N(\varepsilon)$ such that

$$\varepsilon > \int_n^\infty \frac{f\left(\frac{1}{v}\right)}{v^2} dv \geq \int_n^{2n} \frac{f\left(\frac{1}{v}\right)}{v^2} dv \geq n \frac{f\left(\frac{1}{n}\right)}{(2n)^2} = \frac{1}{4} \frac{f\left(\frac{1}{n}\right)}{n},$$

for $n > N$, this implying

$$(11) \quad f\left(\frac{1}{n}\right) = o(n).$$

On the other hand,

$$\int_{\frac{1}{n}}^1 f(t) dt = \frac{1}{n} \int_1^n f\left(\frac{u}{n}\right) du;$$

therefore, using (10) and (11), we obtain

$$\sum_{k=1}^n f\left(\frac{k}{n}\right) = n \int_{\frac{1}{n}}^1 f(t) dt + \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) - \int_1^n f\left(\frac{u}{n}\right) du \right) = n \int_{\frac{1}{n}}^1 f(t) dt + o(n)$$

and so Theorem 3 implies our assertion.

We see that, for the validity of (9), the function $f(t)$ must not necessarily be bounded for $0 \leq t \leq 1$.

2. Case of $f(t)$ having a derivative of first or higher order.

Connection with Riemann's hypothesis.

Suppose that $f(t)$ has a bounded derivative in the interval $0 \leq t \leq 1$, then, applying the mean value theorem of the differential calculus, we can write

$$(12) \quad \sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t) dt = n \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f\left(\frac{k}{n}\right) - f(t) \right) dt =$$

$$= n O \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - t \right) dt = O \left(n \sum_{k=1}^n \frac{1}{n^2} \right) = O(1).$$

Hence it follows at once for the remainder (cf. (4), (6))

$$\sum_{v=1}^{\Phi(x)} f(q_v) - \Phi(x) \int_0^1 f(t) dt = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t) dt \right) =$$

$$= O \sum_{n=1}^{[x]} \left| M\left(\frac{x}{n}\right) \right| = O(x \log x).$$

This result may be easily improved by using the trivial relation⁹⁾

$$(13) \quad \sum_{\nu=1}^{\Phi(x)} \left(\varrho_{\nu} - \frac{\nu}{\Phi} \right)^2 = O(1).$$

We get namely, by (12),

$$\begin{aligned} \sum_{\nu=1}^{\Phi(x)} f(\varrho_{\nu}) - \Phi(x) \int_0^1 f(t) dt &= \\ &= \sum_{\nu=1}^{\Phi(x)} \left(f(\varrho_{\nu}) - f\left(\frac{\nu}{\Phi}\right) \right) + \left(\sum_{\nu=1}^{\Phi(x)} f\left(\frac{\nu}{\Phi}\right) - \Phi(x) \int_0^1 f(t) dt \right) = \\ (14) \quad &= O \sum_{\nu=1}^{\Phi(x)} \left| \varrho_{\nu} - \frac{\nu}{\Phi} \right| + O(1) = O \left(\sqrt{\Phi(x) \sum_{\nu=1}^{\Phi(x)} \left(\varrho_{\nu} - \frac{\nu}{\Phi} \right)^2} \right) + O(1) = \\ &= O \left(x \sqrt{\sum_{\nu=1}^{\Phi(x)} \left(\varrho_{\nu} - \frac{\nu}{\Phi} \right)^2} \right) + O(1), \end{aligned}$$

so that (13) furnishes immediately

$$(15) \quad \sum_{\nu=1}^{\Phi(x)} f(\varrho_{\nu}) - \Phi(x) \int_0^1 f(t) dt = O(x).$$

(13) represented hitherto, to our knowledge, the sharpest (positive) result concerning the order of FRANEL's sum. In a recent paper¹⁰⁾ we have deduced on the basis of the so-called FRANEL identity¹¹⁾

$$(16) \quad \sum_{\nu=1}^{\Phi(x)} \left(\varrho_{\nu} - \frac{\nu}{\Phi} \right)^2 = \frac{1}{12 \Phi(x)} \left\{ \sum_{a=1}^{[x]} \sum_{b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a, b)^2}{ab} - 1 \right\}$$

the much better relation

$$(17) \quad \sum_{\nu=1}^{\Phi(x)} \left(\varrho_{\nu} - \frac{\nu}{\Phi} \right)^2 = O(e^{-c_0(\log x)^{\gamma}}),$$

by using N. TCHUDAKOV's result¹²⁾ on the error term of the prime number theorem

$$(18) \quad \pi(x) - \int_2^x \frac{du}{\log u} = O(x e^{-c_1(\log x)^{\gamma}}).$$

More precisely we use its analogue for the MÖBIUS function¹³⁾

$$(19) \quad M(x) = O(x e^{-c_2(\log x)^{\gamma}}).$$

⁹⁾ See LANDAU [3], or [2], vol. II, 176.

¹⁰⁾ MIKOLÁS [1].

¹¹⁾ FRANEL [1], LANDAU [2], vol. II, 173.

¹²⁾ TCHUDAKOV [1], 591–602.

¹³⁾ See FOGELS [1].

(Here γ denotes any constant between $\frac{1}{2}$ and $\frac{11}{21}$ excl., while c_1, c_2, c_3 are positive constants depending on the choice of γ only.)

On the other hand, in case of the validity of RIEMANN's hypothesis, we have the well-known relations, the first implying the second,

$$(20) \quad M(x) = O\left(x^{\frac{1}{2} + c_4 \frac{\log \log \log x}{\log \log x}}\right),$$

$$(21) \quad \sum_{\nu=1}^{\Phi(x)} \left(\varrho_\nu - \frac{\nu}{\Phi}\right)^2 = O\left(x^{-1 + c_5 \frac{\log \log \log x}{\log \log x}}\right),$$

where c_4, c_5 denote other positive constants¹⁴.

Thus, applying (17) and (21), we obtain at once from (14)

Theorem 3. *Let $f(t)$ be a function having a bounded derivative in the interval $0 \leq t \leq 1$. Then we have the relation*

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) = \Phi(x) \int_0^1 f(t) dt + O(xe^{-c_3(\log x)^\gamma})$$

with $\frac{1}{2} < \gamma < \frac{11}{21}$, $c_3 = c_3(\gamma) > 0$.

If RIEMANN's hypothesis is true, we have besides

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) = \Phi(x) \int_0^1 f(t) dt + O\left(x^{\frac{1}{2} + c_6 \frac{\log \log \log x}{\log \log x}}\right),$$

and therefore

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) = \Phi(x) \int_0^1 f(t) dt + O\left(x^{\frac{1}{2} + \varepsilon}\right)$$

for every $\varepsilon > 0$.

All these relations may be immediately deduced on the basis of the identities (4), without using FRANEL's sum $\sum \left(\varrho_\nu - \frac{\nu}{\Phi}\right)^2$, if we suppose that $f'(t)$ exists and is continuous for $0 \leq t \leq 1$. In this case we must only apply the EULER—MACLAURIN sum-formula in its simplest form:

$$(22) \quad \sum_{k=1}^n g(k) = \int_0^n g(u) du + \frac{1}{2} (g(n) + g(0)) + \int_0^n \left(u - [u] - \frac{1}{2}\right) g'(u) du$$

for the function $g(u) = f\left(\frac{u}{n}\right)$; we thus obtain (cf. (4), (5), (6))

¹⁴ See e. g. LANDAU [2], vol. II, 161—166, 176—177.

$$(23) \quad \sum_{v=1}^{\Phi(x)} f(\varrho_v) = \Phi(x) \int_0^1 f(t) dt + \frac{1}{2} (f(1) - f(0)) + \\ + \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) \int_0^1 \left(nt - [nt] - \frac{1}{2}\right) f'(t) dt.$$

Since for any $a, b^{15)}$

$$(24) \quad \int_0^1 \left(at - [at] - \frac{1}{2}\right) \left(bt - [bt] - \frac{1}{2}\right) dt = \frac{(a, b)^2}{12ab},$$

we obtain by inverting the order of summation and integration, and by applying the inequality of SCHWARTZ, that the last term under (23) may be replaced by the positive square root of

$$(25) \quad \frac{\vartheta}{12} \int_0^1 (f'(t))^2 dt \cdot \sum_{a, b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a, b)^2}{ab} \quad (0 \leq \vartheta = \vartheta(x) \leq 1),$$

which, according to (16), is equal to

$$(26) \quad \vartheta \int_0^1 (f'(t))^2 dt \cdot \left\{ \frac{1}{12} + \Phi(x) \sum_{v=1}^{\Phi(x)} \left(\varrho_v - \frac{v}{\Phi} \right)^2 \right\}.$$

The form (25) renders possible, by (19) and (20), an immediate estimation of the remainder

$$\sum_{v=1}^{\Phi(x)} f(\varrho_v) - \Phi(x) \int_0^1 f(t) dt,$$

while (26) shows the connection with FRANEL's theorem.

For a function $g(u)$ which has continuous derivatives $g'(u), g''(u), \dots, g^{(2r+1)}(u)$ ($r \geq 0$) in the interval $1 \leq u \leq n$, the general form of the EULER-MACLAURIN sum-formula¹⁶⁾

$$(27) \quad \sum_{k=1}^n g(k) = \int_0^n g(u) du + \frac{1}{2} (g(n) + g(0)) + \\ + \sum_{i=1}^r \frac{B_{2i}}{(2i)!} (g^{(2i-1)}(n) - g^{(2i-1)}(0)) + \int_0^n P_{2r+1}(u) g^{(2r+1)}(u) du$$

is valid, where B_2, B_4, \dots are the so-called Bernoullian numbers¹⁷⁾, defined by

¹⁵⁾ LANDAU [2], vol. II, 170.

¹⁶⁾ See e. g. KNOPP [1] 542.

¹⁷⁾ We have $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = B_8 = B_{10} = \dots = 0$.

$$\binom{m}{0} B_0 + \binom{m}{1} B_1 + \dots + \binom{m}{m-1} B_{m-1} = 0 \quad (m = 1, 2, 3, \dots)$$

while the functions with period 1

$$(28) \quad P_{2r+1}(u) = (-1)^{r-1} \sum_{k=1}^{\infty} \frac{2 \sin 2k\pi u}{(2k\pi)^{2r+1}} \quad (r = 0, 1, 2, \dots)$$

are identical for $0 \leq u < 1$ to the corresponding Bernoullian polynomials.

Taking again $g(u) = f\left(\frac{u}{n}\right)$ in (27), and using the identities (4), (5), (6) respectively, we obtain after simple integral-transformations

$$(29) \quad \begin{aligned} \sum_{v=1}^{\Phi(x)} f(\varrho_v) &= \Phi(x) \int_0^1 f(t) dt + \frac{1}{2} (f(1) - f(0)) + \\ &+ \frac{B_2}{2!} (f'(1) - f'(0)) \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right) + \\ &+ \frac{B_4}{4!} (f'''(1) - f'''(0)) \sum_{n=1}^{[x]} \frac{1}{n^3} M\left(\frac{x}{n}\right) + \dots + \\ &+ \frac{B_{2r}}{(2r)!} (f^{(2r-1)}(1) - f^{(2r-1)}(0)) \sum_{n=1}^{[x]} \frac{1}{n^{2r-1}} M\left(\frac{x}{n}\right) + \\ &+ \sum_{n=1}^{[x]} \frac{1}{n^{2r}} M\left(\frac{x}{n}\right) \int_0^1 P_{2r+1}(nt) \cdot f^{(2r+1)}(t) dt, \end{aligned}$$

supposing, of course, that each derivatives of $f(t)$ which occur here exist, and that $f^{(2r+1)}(t)$ is continuous for $0 \leq t \leq 1$.

To deduce another form for the last (remainder) term under (29) we use

Lemma 5. *Let λ be a real number not less than 1. Then, taking*

$$p_\lambda(u) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi u}{n^\lambda},$$

we have

$$\int_0^1 p_\lambda(at) p_\lambda(bt) dt = \frac{\zeta(2\lambda)}{2} \cdot \frac{(a, b)^{2\lambda}}{a^\lambda b^\lambda}.$$

Proof: It follows from PARSEVAL's theorem, using also the fact that positive solutions of the Diophantine equation $au = bv$ (for fixed a, b) are

$$u = k \frac{b}{(a, b)}, \quad v = k \frac{a}{(a, b)} \quad (k = 1, 2, \dots).$$

Now, applying Lemma 5 and the inequality of SCHWARTZ, we obtain (cf. (28))

$$\begin{aligned}
 U_r^2(x) &= \left\{ \int_0^1 f^{(2r+1)}(t) \cdot \left(\sum_{n=1}^{[x]} \frac{1}{n^{2r}} M\left(\frac{x}{n}\right) P_{2r+1}(nt) \right) dt \right\}^2 \leq \\
 &\leq I_r^2 \cdot \int_0^1 \left\{ \sum_{n=1}^{[x]} \frac{1}{n^{2r}} M\left(\frac{x}{n}\right) P_{2r+1}(nt) \right\}^2 dt = \\
 &= I_r^2 \cdot \sum_{a=1}^{[x]} \sum_{b=1}^{[x]} \frac{1}{a^{2r} b^{2r}} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \cdot \int_0^1 P_{2r+1}(at) P_{2r+1}(bt) dt = \\
 &= I_r^2 \cdot \frac{2\zeta(4r+2)}{(2\pi)^{4r+2}} \sum_{a,b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a,b)^{4r+2}}{a^{4r+1} b^{4r+1}},
 \end{aligned}$$

where

$$I_r^2 = \int_0^1 (f^{(2r+1)}(t))^2 dt.$$

Considering that, for positive integral values of λ ,

$$\zeta(2\lambda) = \sum_{n=1}^{\infty} \frac{1}{n^{2\lambda}} = (-1)^{\lambda-1} \frac{B_{2\lambda}(2\pi)^{2\lambda}}{2(2\lambda)!},$$

we see that the square of the remainder term under (29) may be written in the form

$$(31) \quad U_r^2(x) = \vartheta \frac{B_{4r+2}}{(4r+2)!} \cdot \int_0^1 (f^{(2r+1)}(t))^2 dt \cdot \sum_{a,b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a,b)^{4r+2}}{a^{4r+1} b^{4r+1}},$$

with $0 \leq \vartheta = \vartheta(x, r) \leq 1$, which is an immediate generalization of the expressions (25) and (26) respectively.

We draw the attention to the well-estimable sums depending upon x only, which occur on the right-hand side of (29) so to say as weights; in what follows this fact makes mainly the formula useful. If $f(t)$ is a polynomial of degree $2r$ or $2r+1$, $U_r(x)$ vanishes; for example, taking $f(t) \equiv t, t^2, t^3$ resp. we obtain

$$(32) \quad \sum_{\nu=1}^{\Phi(x)} \varrho_{\nu} = \frac{1}{2} \Phi(x) + \frac{1}{2},$$

$$(33) \quad \sum_{\nu=1}^{\Phi(x)} \varrho_{\nu}^2 = \frac{1}{3} \Phi(x) + \frac{1}{2} + \frac{1}{6} \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right),$$

$$(34) \quad \sum_{\nu=1}^{\Phi(x)} \varrho_{\nu}^3 = \frac{1}{4} \Phi(x) + \frac{1}{2} + \frac{1}{4} \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right).$$

3. The "problem of equivalence".

Suppose that RIEMANN'S hypothesis is true, then, according to Theorem 3, the remainder term $\sum f(\rho_v) - \Phi(x) \int_0^1 f(t) dt$ is $O(x^{\frac{1}{2}+\varepsilon})$ for any function in question (i. e. for $f(t)$ having a bounded derivative in $\langle 0, 1 \rangle$).

Taking $f(t) \equiv \cos 2\pi t$, we get from

$$\sum_{\substack{k \leq n \\ (k, n)=1}} e^{\frac{2k\pi i}{n}} = \sum_{\substack{k \leq n \\ (k, n)=1}} \cos \frac{2k\pi}{n} = \mu(n) \quad ^{18)}$$

the identities

$$\sum f(\rho_v) - \Phi(x) \int_0^1 f(t) dt = \sum f(\rho_v) = \sum_{n=1}^{[x]} \mu(n) = M(x),$$

so that, in virtue of LITTLEWOOD'S theorem, the converse of our above proposition is also true in this case: if $f(t) \equiv \cos 2\pi t$, the relation

$$\sum f(\rho_v) - \Phi(x) \int_0^1 f(t) dt = O(x^{\frac{1}{2}+\varepsilon})$$

implies the validity of RIEMANN'S hypothesis.

On the other hand, it is evident that such a converse proposition does not hold for *all* $f(t)$ in question; for example, if $f(t)$ is a function for which $f(t) = -f(1-t)$ when $0 \leq t \leq 1$, then we have (with regard to $\rho_v = 1 - \rho_{\Phi-v}$; $v = 1, 2, \dots$; $\Phi(x) - 1$)

$$\sum_{v=1}^{\Phi(x)} f(\rho_v) - \Phi(x) \int_0^1 f(t) dt \equiv f(1),$$

independently of RIEMANN'S hypothesis.

Therefore it may be raised the question: which conditions must be satisfied by a function $f(t)$ (having a bounded derivative for $0 \leq t \leq 1$), in order that the following assertion be true: "the relation

$$\sum_{v=1}^{\Phi(x)} f(\rho_v) - \Phi(x) \int_0^1 f(t) dt = O(x^{\frac{1}{2}+\varepsilon})$$

holds for every ε if and only if RIEMANN'S hypothesis is valid".

We shall see that this problem — it may be called "problem of equivalence" since we look for relations which are equivalent to RIEMANN'S

¹⁸⁾ See e. g. LARDAU [1], vol. II, 572–573; [2], vol. I, 188.

hypothesis — is not easy to handle in full generality; we get, however, interesting special results.

In what follows a_n, b_n, c_n denote real numbers, $s = \sigma + i\tau$ is a complex variable, so that $\sigma = \Re(s)$, $\tau = \Im(s)$.

Next we need two well-known propositions from the theory of DIRICHLET series¹⁹).

Lemma 6. If

$$S(x) = \sum_{k=1}^{[x]} a_k = O(x^{\alpha+\varepsilon})$$

for every $\varepsilon > 0$, then the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges in the half-plane $\sigma > \alpha$ and represents here a regular function of s .

Lemma 7. If for $\sigma > \sigma_0$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = f(s) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s} = g(s)$$

are absolutely convergent, then the series with the coefficients

$$c_n = \sum_{\substack{d|n \\ d \leq \frac{n}{\alpha}}} a_d b_{\frac{n}{d}} \quad (n = 1, 2, \dots)$$

converges also absolutely in the half-plane $\sigma > \sigma_0$, and one has here

$$(35) \quad \sum_{n=1}^{\infty} \frac{c_n}{n^s} = f(s) \cdot g(s).$$

Theorem 4. Let $f(t)$ be a function having a bounded derivative in $\langle 0, 1 \rangle$. If the relation

$$(36) \quad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t) dt = O(x^{\frac{1}{2}+\varepsilon})$$

holds for every $\varepsilon > 0$, then

1) the function $F(s)$, defined for $\sigma > 1$ by

$$(37) \quad F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t) dt \right),$$

is regular for $\sigma > \frac{1}{2}$, $s \neq 1$;

2) $\zeta(s)$ cannot vanish in the half-plane $\sigma > \frac{1}{2}$, unless at points where $F(s) = 0$.

¹⁹) See e. g. LANDAU [1], vol. I, 121, 131.

Proof: On the basis of Lemma 1 and of the well-known identities

$$(38) \quad \varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d|n} d \mu\left(\frac{n}{d}\right)$$

we can write

$$\sum_{\substack{k \leq n \\ (k, n)=1}} f\left(\frac{k}{n}\right) - \varphi(n) \int_0^1 f(t) dt = \sum_{d|n} \mu\left(\frac{n}{d}\right) \left(\sum_{k=1}^d f\left(\frac{k}{d}\right) - d \int_0^1 f(t) dt \right)$$

so that (35) gives "formally"

$$(39) \quad \sum_{n=1}^{\infty} \frac{\sum_{\substack{k \leq n \\ (k, n)=1}} f\left(\frac{k}{n}\right) - \varphi(n) \int_0^1 f(t) dt}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t) dt}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right);$$

since the series on the right-hand side are plainly, by (12) and $|\mu(n)| \leq 1$, absolutely convergent for $\sigma > 1$, (39) holds, according to Lemma 7, in this half-plane.

Considering that for $\sigma > 1$

$$(40) \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

we have the equality

$$(41) \quad F(s) = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{\substack{k \leq n \\ (k, n)=1}} f\left(\frac{k}{n}\right) - \varphi(n) \int_0^1 f(t) dt \right).$$

Now suppose that (36) is valid. Then the series on the right-hand side of (41) is convergent and regular for $\sigma > \frac{1}{2}$, by virtue of Lemma 6. On the other hand, as is well-known, $\zeta(s)$ is regular in the whole plane except at $s = 1$.

Thus the function

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{\substack{k \leq n \\ (k, n)=1}} f\left(\frac{k}{n}\right) - \varphi(n) \int_0^1 f(t) dt \right),$$

which represents, according to (41), the analytical continuation of

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t) dt \right) \quad (\sigma > 1),$$

is regular for $\sigma > \frac{1}{2}$, except possibly at $s = 1$; this function vanishes, of course, at all points of the half-plane $\sigma > \frac{1}{2}$, where $\zeta(s) = 0$.

We add an important

Corollary. Let $f(t)$ denote a function having a bounded derivative for $0 \leq t \leq 1$, and such that $F(s)$ is regular and has no zeros for $\sigma > \frac{1}{2}$.

Then the relation (36) involves the validity of RIEMANN's hypothesis.

This result suggests to find a $f(t)$, for which we can show the regularity and not-vanishing of $F(s)$ in the half-plane $\sigma > \frac{1}{2}$. In this direction we make good use of the formulae (33), (29), and the well-known fact that, if $\lambda \geq \frac{1}{2}$,

$$(42) \quad \zeta(s + \lambda) \neq 0$$

for $\sigma \geq \frac{1}{2}$.²⁰⁾

Lemma 8. Let λ be a real number not less than $\frac{1}{2}$. RIEMANN's hypothesis is true if and only if we have for every positive ε

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Proof: 1. The first part of our proposition is a trivial consequence of LITTLEWOOD's theorem: assuming the validity of RIEMANN's hypothesis, we have $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$, and therefore

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon} \sum_{n=1}^{[x]} \frac{1}{n^{\lambda+\frac{1}{2}+\varepsilon}}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

2. Suppose that the relation

$$(43) \quad \sum_{n \leq x} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

holds for every positive ε .

Using Lemma 7 and (40), we find

$$(44) \quad \sum_{n=1}^{\infty} \frac{\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{1}{d^\lambda}}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^\lambda} \right) \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) = \frac{\zeta(s+\lambda)}{\zeta(s)} \quad (\sigma > 1).$$

The series on the left-hand side is, by Lemma 6 and (43) (cf. (1)), convergent and regular for $\sigma > \frac{1}{2}$; as $\zeta(s+\lambda)$ is also regular and has no zeros here, (44) implies that $\zeta(s) \neq 0$ in the half-plane $\sigma > \frac{1}{2}$.

²⁰⁾ This follows from the not-vanishing of $\zeta(s)$ for $\sigma \geq 1$. (See e. g. LANDAU [1], vol. I, 154, 166.)

Theorem 5. 1) For every polynomial of second degree

$$f(t) \equiv a_0 t^2 + a_1 t + a_2 \quad (a_0 \neq 0)$$

the relation (36) is equivalent to RIEMANN's hypothesis.

2) Let

$$f(t) \equiv a_0 t^3 + a_1 t^2 + a_2 t + a_3 \quad (a_0 \neq 0),$$

then, in order that (36) should be equivalent to RIEMANN's hypothesis, it is necessary and sufficient that

$$a_1 \neq -\frac{3}{2} a_0.$$

Corollary. RIEMANN's hypothesis is true if and only if we have for every positive ε

$$\sum_{v=1}^{\Phi(x)} \varrho_v^2 - \frac{\Phi(x)}{3} = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad \text{or} \quad \sum_{v=1}^{\Phi(x)} \varrho_v^2 - \frac{\Phi(x)}{4} = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Proof: Applying the identities (32), (33), (34), we obtain

$$\sum_{v=1}^{\Phi(x)} f(\varrho_v) - \Phi(x) \int_0^1 f(t) dt = \frac{a_0 + a_1}{2} + \frac{a_0}{6} \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right)$$

in the first,

$$\sum_{v=1}^{\Phi(x)} f(\varrho_v) - \Phi(x) \int_0^1 f(t) dt = \frac{a_0 + a_1 + a_2}{2} + \left(\frac{a_0}{4} + \frac{a_1}{6}\right) \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right)$$

in the second case, so that Lemma 8 involves immediately our assertions.

Theorem 6. Whatever be r except 1, there is an infinity of polynomials

$$f(t) \equiv a_0 t^r + a_1 t^{r-1} + \dots + a_{r-1} t + a_r \quad (a_0 \neq 0)$$

such that (36) is equivalent to RIEMANN's hypothesis.

Proof: By the theorem just proved, we may suppose that $r \geq 4$. The application of (28) gives now

$$(45) \quad \sum_{v=1}^{\Phi(x)} f(\varrho_v) - \Phi(x) \int_0^1 f(t) dt = \frac{1}{2} (a_0 + a_1 + \dots + a_{r-1}) + \\ + \frac{B_2}{2} \left(a_0 \binom{r}{1} + a_1 \binom{r-1}{1} + \dots + a_{r-2} \binom{2}{1} \right) \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right) + \\ + \frac{B_4}{4} \left(a_0 \binom{r}{3} + a_1 \binom{r-1}{3} + \dots + a_{r-4} \binom{4}{3} \right) \sum_{n=1}^{[x]} \frac{1}{n^3} M\left(\frac{x}{n}\right) + \dots$$

The last term on the right-hand side is

$$\begin{cases} \frac{B_r}{r} a_0 \binom{r}{r-1} \sum_{n=1}^{[\infty]} \frac{1}{n^{r-1}} M\left(\frac{x}{n}\right) & \text{if } r \text{ is even,} \\ \frac{B_{r-1}}{r-1} \left(a_0 \binom{r}{r-2} + a_1 \binom{r-1}{r-2} \right) \sum_{n=1}^{[\infty]} \frac{1}{n^{r-2}} M\left(\frac{x}{n}\right) & \text{if } r \text{ is odd.} \end{cases}$$

The coefficients a_0, a_1, \dots, a_{r-4} can plainly be chosen so that the system of equations

$$\begin{aligned} \binom{r}{1} + \binom{r-1}{1} \xi_1 + \dots + \binom{4}{1} \xi_{r-4} + \binom{3}{1} \xi_{r-3} + \binom{2}{1} \xi_{r-2} &= 0, \\ \binom{r}{3} + \binom{r-1}{3} \xi_1 + \dots + \binom{4}{3} \xi_{r-4} &= 0, \end{aligned} \quad (46)$$

$$\text{(finally): } \begin{cases} \binom{r}{r-3} + \binom{r-1}{r-3} \xi_1 + \binom{r-2}{r-3} \xi_2 = 0, & \text{if } r \text{ is even,} \\ \binom{r}{r-4} + \binom{r-1}{r-4} \xi_1 + \binom{r-2}{r-4} \xi_2 + \binom{r-3}{r-4} \xi_3 = 0, & \text{if } r \text{ is odd,} \end{cases}$$

where $\xi_k = \frac{a_k}{a_0}$ ($k = 1, 2, \dots, r-2$), should be satisfied; when r is odd, let besides be

$$\xi_1 \neq -\frac{3}{2}.$$

Since we have $\left\lfloor \frac{r}{2} - 1 \right\rfloor$ equations for $(r-2)$ unknowns and $\left\lfloor \frac{r}{2} - 1 \right\rfloor < r-2$ if $r \geq 4$, the number of solutions of (46) is infinite for every degree $r \geq 4$.

If all our conditions are fulfilled, then each term on the right-hand side of (45) except the last vanishes, so that the proposition follows at once from Lemma 8.

Theorem 7. *Let $f(t)$ be a function such that $f'(t), f''(t), f'''(t) \neq 0$ exist, $f'''(t)$ is continuous for $0 \leq t \leq 1$, and that the condition*

$$\frac{|f'(1) - f'(0)|}{\int_0^1 |f'''(t)| dt} > \frac{3\zeta(3)}{2\pi} = 0.574 \dots$$

is fulfilled. Then (36) is equivalent to RIEMANN's hypothesis.

Proof: Suppose that $f(t)$ is a function satisfying our conditions.

1. If RIEMANN's hypothesis is true, then we have, by virtue of Theorem 5,

$$\sum_{v=1}^{\Phi(x)} f(\rho_v) = \Phi(x) \int_0^1 f(t) dt + O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad (\varepsilon > 0).$$

2. Assume that the relation (36) holds (for every positive ε). The use of (22) (cf. (30)) shows that

$$\begin{aligned}\sum_{k=1}^n f\left(\frac{k}{n}\right) &= \int_0^n f\left(\frac{u}{n}\right) du + \frac{1}{2}(f(1)-f(0)) + \frac{1}{n} \int_0^n P_1(u) f'\left(\frac{u}{n}\right) du = \\ &= n \int_0^1 f(t) dt + \frac{1}{2}(f(1)-f(0)) + \int_0^1 P_1(nt) f'(t) dt,\end{aligned}$$

and so, by (39) and (40), we can write for $\sigma > 1$

$$\begin{aligned}(47) \quad \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^1 P_1(nt) f'(t) dt &= \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{\substack{k \leq n \\ (k, n)=1}} f\left(\frac{k}{n}\right) - \varphi(n) \int_0^1 f(t) dt \right) - \frac{1}{2}(f(1)-f(0)).\end{aligned}$$

Consider the function with the period 1

$$(48) \quad P_2(u) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos 2k\pi u}{(k\pi)^2};$$

it is continuous everywhere, and we have plainly

$$P_2'(u) = P_1(u)$$

if u is not an integer.

Now, the series on the right-hand side of (47) is, by hypothesis (36), regular for $\sigma > \frac{1}{2}$ (cf. Lemma 6.); the series on the left (converges and so) is regular in the half-plane $\sigma > 0$ by

$$\begin{aligned}(49) \quad \int_0^1 P_1(nt) f'(t) dt &= \frac{1}{12n} (f'(1) - f'(0)) - \frac{1}{n} \int_0^1 P_2(nt) f''(t) dt \leq \\ &\leq \frac{1}{12n} (|f'(1) - f'(0)| + \int_0^1 |f''(t)| dt) = O\left(\frac{1}{n}\right).\end{aligned}$$

Consequently, if we have for $\sigma > \frac{1}{2}$

$$(50) \quad \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^1 P_1(nt) f'(t) dt \neq 0,$$

then it follows that $\zeta(s) \neq 0$ in this half-plane.

To verify (50), we write using (49)

$$(51) \quad \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^1 P_1(nt) f'(t) dt = \frac{1}{12} (f'(1) - f'(0)) \zeta(s+1) - \\ - \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \int_0^1 P_2(nt) f''(t) dt = \zeta(s+1) \left\{ \frac{1}{12} (f'(1) - f'(0)) - \sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}} \right\}.$$

Here the coefficients b_n can be easily determined by the condition (cf. (40)).

$$\sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+1}} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \int_0^1 P_2(nt) f''(t) dt \right);$$

we have namely, by virtue of Lemma 7,

$$b_n = \sum_{\delta|n} \mu\left(\frac{n}{\delta}\right) \int_0^1 P_2(\delta t) f''(t) dt.$$

(Both series on the right-hand side, and so their product too, converge absolutely for $\sigma > 0$.)

Partial integration shows that (cf. (28))

$$\int_0^1 P_2(nt) f''(t) dt = -\frac{1}{n} \int_0^1 P_3(nt) f'''(t) dt,$$

and, applying (2), we get

$$B(u) = \sum_{n=1}^{[u]} |b_n| \leq \sum_{n=1}^{[u]} \sum_{\delta|n} \left| \int_0^1 P_2(\delta t) f''(t) dt \right| = \\ = \sum_{n=1}^{[u]} \left[\frac{u}{n} \right] \cdot \frac{1}{n} \left| \int_0^1 P_3(nt) f'''(t) dt \right| \leq u \sum_{n=1}^{[u]} \frac{1}{n^2} \int_0^1 |P_3(nt)| |f'''(t)| dt < \\ < u \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{k=1}^{\infty} \frac{2}{(2k\pi)^3} \right) \cdot \int_0^1 |f'''(t)| dt = \frac{\zeta(3)}{24\pi} u \cdot \int_0^1 |f'''(t)| dt.$$

On the other hand, by

$$\sum_{n=1}^{[v]} \frac{|b_n|}{n^{\sigma+1}} = \sum_{n=1}^{[v]} B(n) \left(\frac{1}{n^{\sigma+1}} - \frac{1}{(n+1)^{\sigma+1}} \right) + \frac{B(v)}{([v]+1)^{\sigma+1}},$$

and $B(v) = O(v)$, it follows for $\sigma > 0$

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n^{\sigma+1}} = \sum_{n=1}^{\infty} B(n) \cdot (\sigma+1) \int_n^{n+1} \frac{du}{u^{\sigma+2}} = (\sigma+1) \int_1^{\infty} \frac{B(u)}{u^{\sigma+2}} du,$$

so that we can write

$$\begin{aligned}
 \left| \sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}} \right| &\leq \sum_{n=1}^{\infty} \frac{|b_n|}{n^{s+1}} = \\
 (52) \quad &= (\sigma+1) \int_1^{\infty} \frac{B(u)}{u^{\sigma+2}} du < (\sigma+1) \frac{\zeta(3)}{24\pi} \left(\int_0^1 |f'''(t)| dt \right) \cdot \int_1^{\infty} \frac{du}{u^{\sigma+1}} = \\
 &= \left(1 + \frac{1}{\sigma} \right) \frac{\zeta(3)}{24\pi} \int_0^1 |f'''(t)| dt \quad (\sigma > 0).
 \end{aligned}$$

Finally, using (51) and (52), we obtain for $\sigma > \frac{1}{2}$ (cf. (42))

$$\begin{aligned}
 \left| \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^1 P_1(nt) f'(t) dt \right| &\geq |\zeta(s+1)| \left(\frac{1}{12} |f'(1) - f'(0)| - \left| \sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}} \right| \right) > \\
 &> \frac{1}{12} |\zeta(s+1)| \left(|f'(1) - f'(0)| - \left(1 + \frac{1}{\sigma} \right) \frac{\zeta(3)}{2\pi} \int_0^1 |f'''(t)| dt \right) > \\
 &> \frac{1}{12} |\zeta(s+1)| \left(|f'(1) - f'(0)| - \frac{3\zeta(3)}{2\pi} \int_0^1 |f'''(t)| dt \right) > 0,
 \end{aligned}$$

which proves (50), and thus our Theorem.

It is easy to find such functions $f(t)$ for which the conditions of the proposition just proved are fulfilled; thus, considering the cases

$$f(t) \equiv e^{\lambda t} \quad \left(\lambda \neq 0, |\lambda| < \frac{2\pi}{3\zeta(3)} = 1.74 \dots \right)$$

and

$$f(t) \equiv \cos \lambda t \quad \left(0 < \lambda \leq \frac{\pi}{2} \right),$$

we obtain the

Corollary. The relations (holding for every $\varepsilon > 0$)

$$(53) \quad \sum_{v=1}^{\Phi(x)} e^{\lambda \varrho_v} - \frac{e^{\lambda} - 1}{\lambda} \Phi(x) = O\left(x^{\frac{1}{2} + \varepsilon}\right) \quad \left(\lambda \neq 0, |\lambda| > \frac{2\pi}{3\zeta(3)} = 1.74 \dots \right)$$

and

$$(54) \quad \sum_{v=1}^{\Phi(x)} \cos \lambda \varrho_v - \frac{\sin \lambda}{\lambda} \Phi(x) = O\left(x^{\frac{1}{2} + \varepsilon}\right) \quad \left(0 < \lambda \leq \frac{\pi}{2} \right)$$

are equivalent to RIEMANN's hypothesis.

In case of $f(t) \equiv \cos 2\pi t$, implying LITTLEWOOD's theorem, our conditions are not satisfied, for

$$f'(1) - f'(0) = 0,$$

but we have

$$F(s) \equiv 1 \neq 0$$

in Theorem 4, so that our Corollary to Theorem 4 and Theorem 3 involve immediately the proposition in question.

By using special properties of $e^{\lambda t}$, $\cos \lambda t$, the above conditions for λ may be improved to

$$|\lambda| < 2 \sqrt{\frac{5}{\zeta(3)}} = 4.078 \dots, \lambda \neq 0, \quad ((53))$$

$$|\lambda| < 2 \sqrt{\frac{5}{\zeta(3) + \frac{5}{\pi^2}}} = 3.432 \dots, \lambda \neq 0, \pm \pi. \quad ((54))$$

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²¹⁾ See MIKOLÁS [2].